Absolute Optimal Solution For a Compact and Convex Game

Rabia Nessah
IESEG School of Management, LEM-CNRS (UMR 8179)
Abstract

This paper investigates the existence of absolute optimal solutions for a partition $P$ in continuous and quasiconcave games. We show that the $P$-consistence property introduced in the paper, together with the quasiconcavity and continuity of payoffs, permits the existence of $P$-absolute optimal solutions in games with compact and convex strategy spaces. The $P$-consistency property is a general condition that cannot be dispensed with for the existence of $P$-absolute optimal solutions. We also characterize the existence of $P$-absolute optimal solutions by providing necessary and sufficient conditions. Moreover, we suggest an algorithm for efficiently computing $P$-absolute optimal solutions.

Keywords: $n$-Person Game, Multiple objectives Game, Strong Equilibrium, Absolute Optimal Solution.
1 Introduction

In 1959, Aumann [1959] introduced the strong Nash equilibrium (SNE) which ensures a more restrictive stability than the Nash equilibrium one (Nash [1951]). A SNE is a Nash equilibrium such that there is a nonempty set of players who could all gain by deviating together to some other combination of strategies that is jointly feasible for them, when the other players who are not in this set are expected to stay with their equilibrium strategies (see also Bernheim et al. [1987]). Since this requirement applies to the grand coalition of all players, SNE must be Pareto efficient. So, a SNE is not only immune to unilateral deviations, but also to deviations by coalitions.

We know that the class of games for which a SNE exists is not large and the existence conditions are very restrictive (Ichiishi [1981]). In order to avoid these problems, we consider a hybrid game in the sense of Zhao [1992]. This game is played in two stages. First, players form coalitions, then these coalitions, as super-players play non-cooperatively against each other.

Zhao [1991, 1992] introduced an intermediate solution concept which is between the cooperative and noncooperative solutions of a multiple objectives game. The cooperation in the first part consists of the definition of a partition. The second stage, an equilibrium solution is defined relative to this partition (see definitions 2.7, 2.8). The same author gives a sufficient condition for the existence of a multiple objectives game to have hybrid and quasi-hybrid solutions for any partition. In this framework, firstly $\tilde{x}$ is a hybrid solution if for a given coalition $K$, no sub-coalition $q$ of $K$ has any interest to deviate from this equilibrium (Definition of core), because this deviation is penalized if the remaining of coalition $K$ chooses his strategy $\tilde{x}_{-q}$ in the same equilibrium. Secondly, $\tilde{x}$ is a quasi-hybrid solution if for each given coalition $K$, $\tilde{x}_K$ is properly Pareto efficient for the multiple objectives sub-game relatively to $K$. Zhao [1999-a, 1999-b] defines $\alpha$-hybrid and $\beta$-hybrid solutions, respectively, and gives the sufficient conditions for their existence.

This paper investigates the existence of absolute optimal solutions for an $n$-person game and weighted absolute optimal solution for a multiple objectives game. In particular, $\tilde{x}$ is an absolute optimal solution relative to a partition $P$ if no player in any coalition $S$ in $P$, can be better off when the players deviate from their $P$-absolute optimal strategy profile $\tilde{x}_S$. This means that at $P$-absolute optimal solution, the players in coalition $S \in P$ play a strategy profile that maximizes the payoff of their players. This equilibrium is stable against deviation of any player from $S \in P$. A strategy profile $\tilde{x}$ is a weighted absolute optimal solution relative to a partition $P$ if no player in any coalition $S$ in $P$, can be better off when the players deviate from their $P$-weighted absolute optimal strategy profile $\tilde{x}_S$. This means that at $P$-weighted absolute optimal, the players in coalition $S$ play a strategy profile that maximizes the weighted payoff of their players. Notice that any $P$-absolute optimal solution is both a hybrid and a quasi-hybrid solution. In this paper we fill this gap by proposing some existence results on absolute optimal and weighted absolute optimal solutions.
relative to a partition of players $P$ in general $n$-person and multiple objectives games, respectively. We show that the $P$-consistence property and $P$-weighted consistence property introduced in the paper, together with the quasiconcavity and continuity of payoffs, permit the existence of $P$-absolute optimal, and $P$-weighted absolute optimal solutions in games with compact and convex strategy spaces. The consistence property is a general condition that cannot be dispensed with for the existence of $P$-absolute optimal, and $P$-weighted absolute optimal solutions, which requires that for each $x \in X$, there exists $z \in X$ such that for each coalition $S \in P$, $z_S$ is an element of the best-reply correspondence formed by the coalition $S$. The $P$-consistence property is relatively easy to check, say, by using the same methods as for finding the maximum of a utilitarian social welfare function for every component $z_{h,S}$ of $z_{j1}^j$ that is equal to $z_{h,S}$ of $z_{j2}^j$ obtained for each $j_1, j_2 \in S$. We also characterize the existence of absolute optimal solution relative to a partition of players $P$ by providing necessary and sufficient conditions. Moreover, we suggest an algorithm that can be used to efficiently compute an absolute optimal solution.

The remainder of the paper is organized as follows. Section 2 presents the notions, definitions, and some properties. Section 3 establishes sufficient conditions for the existence of an absolute optimal solution and provides a method for its computation. Section 4 concludes.

2 Absolute Optimal Solution and Its Properties

Consider the following $n$-person game in normal form defined as

$$G = \langle I, X_i, u_i \rangle$$

(2.1)

where $I = \{1, ..., n\}$ is the set of players, $X = \prod_{i \in I} X_i$ is the set of strategies of the game, where $X_i$ (a nonempty set is in $\mathbb{R}^{l(i)}$) is the set of strategies of player $i$, and $u_i$ is $i$'s payoff function $X \to \mathbb{R}$. Let $\mathfrak{S}$ denote the set of all coalitions (i.e., nonempty subsets of $I$). For each coalition $S \in \mathfrak{S}$, denote by $-S = \{i \in I : i \notin S\}$ the complement of coalition $S$. If $S$ is reduced to a singleton $\{i\}$, we denote simply by $-i$ all other players except player $i$. We also denote by $X_S = \prod_{i \in S} X_i$ the set of strategies of players in coalition $S$. If $\{K_j\}_{j \in \{1, ..., s\}}$ is a partition of $I$, any strategy profile $x = (x_1, ..., x_n) \in X$ then can be written as $x = (x_{K_1}, x_{K_2}, ..., x_{K_s})$ with $x_{K_i} \in X_{K_i}$.

We say that a game $G$ (2.1) is compact, convex, and continuous, respectively if, for all $i \in I$, $X_i$ is compact and convex, and $u_i$ is and continuous on $X$, respectively.

For any two vectors $a, b \in \mathbb{R}^n$, $a \succeq b$ if and only if $a_i \geq b_i$ for all $i = 1, ..., n$, $a \succ b$ if and only if $a \succeq b$ and $a \neq b$; and finally $a \gg b$ if and only if $a_i > b_i$ for all $i = 1, ..., n$.

A general multiple objectives mathematical programming problem is defined as:

**Vector Maximization (VM):**

$$\max_{z \in \mathbb{Z}} F(x) = \{f_1(x), ..., f_m(x)\}, \quad m > 1,$$

(2.2)
where \( Z \) is the feasible strategy set, which is assumed to be a nonempty subset, \( f_i : Z \to \mathbb{R}, \ i = 1, \ldots, m \), are the objective functions to be maximized.

**Definition 2.1** A strategy profile \( \pi \in Z \) is said to be weakly Pareto efficient (WPE) for VM (2.2) if there does not exist \( y \in Z \) such that \( F(y) \gg F(\pi) \).

**Definition 2.2** A strategy profile \( \pi \in Z \) is said to be Pareto efficient (PE) for VM (2.2) if there does not exist \( y \in Z \) such that \( F(y) \succ F(\pi) \).

**Definition 2.3** A strategy profile \( \pi \in Z \) is said to be properly Pareto efficient (PPE) for VM (2.2) if \( \pi \in R^*e \) and if there is \( L > 0 \) such that for any \( i \) and \( x \in Z \) satisfying \( f_i(x) > f_i(\pi) \), there is a \( j \neq i \) so as \( f_j(x) < f_j(\pi) \) and \( \frac{f_i(x) - f_i(\pi)}{f_j(x) - f_j(\pi)} \leq L \).

Now, let us consider the following notation. Let \( S \) be any nonempty coalition in \( \mathcal{S} \), denote by \( \#S \) the number of elements in \( S \). Let \( x_S = (x_i : i \in S) \) and \( x_{-S} = (x_i : i \notin S) \) be the strategy of coalition \( S \) and \( -S \), respectively.

\( u_S(\cdot) = (u_i(\cdot) : i \in S) \in \mathbb{R}^{\#S} \) be the vector payoff of coalition \( S \). Zhao [1992] defines the following worst vector payoffs to coalition \( S \) as follows:

\[ u_S(x_S) = (u_S^i(x_S) : i \in S) \]

and by

\[ u_S^i(x_S) = \{ \inf_{y_{-S} \in X_{-S}} u_j^i(x_S, y_{-S}) : j = 1, \ldots, m(i) \} \in \mathbb{R}. \]

The cooperative and non-cooperative solutions are then defined as:

**Definition 2.4** A strategy profile \( \pi \in X \) is said to be Nash equilibrium (NE) of game G (2.1), if for each \( i \in I, \pi_i \in X_i \) is a solution to the following maximization problem:

\[ \max_{x_i \in X_i} u_i(x_i, \pi_{-i}). \]

That is, \( \pi \) is a Nash equilibrium if each player \( i \) chooses a \( \pi_i \) as a best response to all others’ strategies \( \pi_{-i} \).

**Definition 2.5** A strategy profile \( \pi \in X \) is said to be properly strong Nash equilibrium (SNE) of game G (2.1), if for each \( S \in \mathcal{S}, \pi_S \in X_S \) is properly Pareto efficient of the vector maximization problem:

\[ \text{VM}_S(\pi_{-S}) : \left\{ \max_{x_S \in X_S} u_S(x_S, \pi_{-S}) = [u_i(x_S, \pi_{-S}) : i \in S] \right\}. \]
A strategy profile is a properly strong Nash equilibrium if no coalition (including the grand coalition, i.e., all players collectively) can profitably deviate from the prescribed profile. This definition immediately implies that any properly strong equilibrium is both properly Pareto efficient and a Nash equilibrium. This equilibrium is stable with regard to the deviation of any coalition.

**Remark 2.1** If we replace the properly Pareto efficiency by Pareto efficiency (or weakly Pareto efficient) in Definition 2.5, then π is said to be a *super strong Nash equilibrium* (strong Nash equilibrium).

**Definition 2.6** A strategy profile $\pi \in X$ is a *core solution* of game $G$ (2.1), if for each $S \in \mathcal{S}$, we have

$$\Omega_S(\pi) = \{ x_S \in X_S : \overline{u}_S(x_S) \geq u(\pi)_S \} = \emptyset,$$

where $u(\pi)_S = (u_i(\pi) : i \in S)$.

$\pi$ is a core solution if no coalition $S$ can, by choosing another strategy available to $S$, guarantee a higher payoff for each of its members independently of the actions of the outside players.

Zhao introduced the following concepts.

**Definition 2.7** (Zhao [1999-a]) Let $P = \{P_i\}_{i \in M}$ be a partition of $I$. A feasible strategy profile $\pi \in X$ is said to be an $\alpha$-hybrid solution\(^1\) ($\alpha$-HS) for the game $G$ (2.1) if for each coalition $S \in P$, $\pi_S$ is in the $\alpha$-core\(^2\) of the vector maximization problem:

$$\text{VM}_S(\pi_{-S}) : \left\{ \max_{x_S \in X_S} u_S(x_S, \pi_{-S}) = [u_i(x_S, \pi_{-S}), i \in S] \right\}.$$  

**Definition 2.8** (Zhao [1999-b]) Let $P = \{P_i\}_{i \in M}$ be a partition of $I$. A feasible strategy profile $\pi \in X$ is said to be a $\beta$-hybrid solution\(^3\) ($\beta$-HS) for the game $G$ (2.1) if for each coalition $S \in P$, $\pi_S$ is in the $\beta$-core\(^4\) of the vector maximization problem:

$$\text{VM}_S(\pi_{-S}) : \left\{ \max_{x_S \in X_S} u_S(x_S, \pi_{-S}) = [u_i(x_S, \pi_{-S}), i \in S] \right\}.$$  

\(^1\)The Weakly $\alpha$-Core: A strategy profile $\pi \in X$ is in the weakly $\alpha$-core of a game $G = \langle I, X, u \rangle$, if for each $S \in \mathcal{S}$ and for each $x_S \in X_S$, there exists an $y_S \in X_S$ such that $u_i(x_S; y_S) \leq u_i(\pi)$ for at least some $i \in S$.

\(^2\)Let a noncooperative game $G = \langle I, X, u \rangle$, then the $\alpha$-cooperative game corresponding to $G$ is defined as follows $\overline{G}_\alpha = \langle I, v_{\alpha}(\cdot) \rangle$ where $v_{\alpha}(S) = \max_{x_S \in X_S, y_S \in X_S} \min_{i \in S} \sum_{j \in S} u_j(x_S, y_S)$.

\(^3\)The Weakly $\beta$-Core: A strategy profile $\pi \in X$ is in the weakly $\beta$-core of a game $G = \langle I, X, u \rangle$, if for each $S \in \mathcal{S}$, there exists a $y_S \in X_S$ such that for each $x_S \in X_S$, $u_i(x_S; y_S) \leq u_i(\pi)$ for at least some $i \in S$.

\(^4\)Let a noncooperative game $G = \langle I, X, u \rangle$, then the $\beta$-cooperative game corresponding to $G$ is defined as follows $\overline{G}_\beta = \langle I, v_{\beta}(\cdot) \rangle$ where $v_{\beta}(S) = \min_{y_S \in X_S, x_S \in X_S} \max_{i \in S} \sum_{j \in S} u_j(x_S, y_S)$.
Following the definitions of Nash, strong Nash, core, $\alpha$-hybrid, $\beta$-hybrid solutions, we introduce the definition of the concept of absolute optimal solution, which is also associated to a partition of players.

**Definition 2.9** Let $P = \{P_i\}_{i \in M}$ be a partition of $I$. A feasible strategy $\pi \in X$ is said to be a $P$-absolute optimal solution (AO) for the game $G$ (2.1) if for each coalition $S \in P$, we have

$$u_S(\pi) \geq u_S(y_S, \pi_{-S}), \forall y_S \in X_S.$$

Or precisely $\pi_S \in \bigcap_{j \in S} \text{Arg} \max \{u_j(y_S, \pi_{-S}), y_S \in X_S\}$ for all $S \in P$.

It is easy to see that when $M = I$, $P = \{\{i\}, i \in I\}$ is a partition of $I$, then a $P$-absolute optimal solution is a Nash equilibrium. A strategy profile $\pi$ is a $P$-absolute optimal solution if no player in any coalition $S$ in $P$, can be better off when the players deviate from their $P$-absolute optimal strategy profile $\pi_S$. This means that at $P$-absolute optimal solution, the players in coalition $S \in P$ play a strategy profile that maximizes the payoff of their players. This equilibrium is stable against deviation of any player of $S \in P$.

**Remark 2.2** The $P$-absolute optimal solution is stable. Indeed, if the complement coalition of $S$ chooses his (her) strategy $\pi_{-S}$ in a $P$-absolute optimal solution, then it constrains all players of $S$ to choose their strategies in the same $P$-absolute optimal solution; if any player $j$ of set $S$ deviates from his strategy $\pi_j$, then this player cannot improve his payoff.

**Remark 2.3** Any $P$-absolute optimal solution is also a Nash equilibrium. In that sense, the $P$-absolute optimal solution can be considered as a refinement of the Nash equilibrium. In case of multiplicity of $P$-absolute optimal solutions, if a coalition chooses publicly a strategy which pertains to a certain $P$-absolute optimal solution, other players will follow and that given $P$-absolute optimal solution will be selected. In contrast, when there is no $P$-absolute optimal solution in the game but multiplicity of Nash equilibria, coalitions need to coordinate in order to select joint strategies leading to a chosen Nash equilibrium.

We can also define a strong Nash equilibrium relatively to partition $P$ as follows.

**Definition 2.10** Let $P = \{P_i\}_{i \in M}$ be a partition of $I$. A strategy profile $\pi \in X$ is said to be $P$-properly strong Nash equilibrium (PSNE) of game $G$ (2.1), if for all $S \in P$, and for each $K \in \mathcal{S}_S$, $\pi_K \in X_K$ is properly Pareto efficient strategy profile of the vector maximization problem:

$$\text{VM}_K(\pi_{-K}) : \left\{ \max_{x_K \in X_K} u_K(x_K, \pi_{-K}) = [u_i(x_K, \pi_{-K}), i \in K] \right\}.$$

\footnote{We can also define the $P$-super-strong Nash equilibrium, $P$-strong Nash equilibrium by replacing the “properly Pareto efficient” by “Pareto efficient”, and “weakly Pareto efficient” in Definition 2.10, respectively.}
where $\mathcal{S}$ is the set of all nonempty coalitions of $S$.

A strategy profile $\pi$ is a $P$-properly strong Nash equilibrium if no sub-coalition $K$ in $S$ (for each $S \in P$) can profitably deviate from the prescribed profile $\pi_K$.

**Property 2.1** Let $P = \{P_i\}_{i \in M}$ be a partition of $I$ and $X_{AO}^P$, $X_{PSNE}^P$, $X_{a}^P$ and $X_{b}^P$ denote the sets of absolute solutions, $P$-strong Nash equilibria, $\alpha$-hybrid and $\beta$-hybrid solutions, respectively, then $X_{AO}^P \subseteq X_{PSNE}^P \subseteq X_{b}^P \subseteq X_{a}^P$.

**Remark 2.4** The $P$-absolute optimal solution or $P$-strong Nash equilibrium does not imply nor is implied by coalition-proof Nash equilibrium in Bernheim et al. [1987] or negotiation-proof Nash equilibrium in Xue [2000].

Let us consider the following example.

**Example 2.1** Consider the three-player game with the following payoff function defined from $[0, 1] \times [0, 1] \times [0, 1]$, $x = (x_1, x_2, x_3)$.

$$
\begin{align*}
    u_1(x) &= x_1 + x_2x_3, \\
    u_2(x) &= x_1 + x_2 - x_3, \\
    u_3(x) &= x_1 - 4x_2 + 3x_3.
\end{align*}
$$

The unique Nash equilibrium of this game is $x = (1, 1, 1)$. This game has not a strong Nash equilibrium. Indeed, let $\bar{x}$ be any strategy in $X$. We distinguish two cases: If $\bar{x} = (1, 1, 1)$, then there exists a coalition $S = \{2, 3\}$ and $y_2 = 0.5, y_3 = 0.4$ such that

$$
\begin{align*}
    u_2(\bar{x}_1, y_2, y_3) &= 1.1 \geq 1 = u_2(\bar{x}); \\
    u_3(\bar{x}_1, y_2, y_3) &= 0.2 > 0 = u_3(\bar{x})
\end{align*}
$$

If $\bar{x} \neq (1, 1, 1)$. Thus there exists $\bar{x}_i \neq 1, i = 1, 2, 3$. Then, there exists $S = \{i\}$ and a strategy $y_i = 1$ such that

$$u_i(y_i, \bar{x}_{-i}) > u_i(\bar{x}).$$

The game considered in this example has a $P$-absolute solution and $P$-strong Nash equilibrium, if $P = \{\{1, 2\}, \{3\}\}$ or $S = \{\{1, 3\}, \{2\}\}$. Indeed, let for example $P = \{\{1, 2\}, \{3\}\}$.

- $S_1 = \{1, 2\}$, arg $\max_{y_1, y_2 \in [0, 1]} u_1(y_1, y_2, 1) = \{(1, 1)\}$ and $\arg \max_{y_1, y_2 \in [0, 1]} u_2(y_1, y_2, 1) = \{(1, 1)\}$.
- $S_2 = \{3\}$, arg $\max_{y_3 \in [0, 1]} u_3(1, 1, y_3) = \{(1)\}$. 


Then, the strategy profile \((1, 1, 1)\) is both \(P\)-absolute solution and \(P\)-strong Nash equilibrium. This example illustrate the situation where strong Nash equilibrium does not exists but \(P\)-absolute solution and \(P\)-strong Nash equilibrium exist \((P = \{\{1, 2\}, \{3\}\} \text{ or } P = \{\{1, 3\}, \{2\}\})\), i.e., the players 1 and 2 or player 1 and 3 can cooperate but players 2 and 3 cannot.

In the case where \(P = \{P_1, P_2\}\) is a partition of \(I\), then the following lemma shows that in a zero-sum game, the \(P\)-absolute solution possesses the average property for any coalition, i.e. the average of payoff functions of players in any coalition has the same value for all \(P\)-absolute optimal solutions.

**Lemma 2.1** Let \(P = \{P_1, P_2\}\) be a partition of \(I\) and the game \(G\) (2.1) be a zero-sum. If \(\bar{x}\) and \(\tilde{x}\) are two different \(P\)-absolute solutions for the game \(G\) (2.1), then for all \(S \in P\), \(\sum_{h \in S} u_h(\bar{x}) = \sum_{h \in S} u_h(\tilde{x})\).

**Proof.** Let \(\bar{x}\) and \(\tilde{x}\) are two different \(S\)-equilibrium for the game \(G\) (2.1). Then for any \(i = 1, 2\), we have

\[
\begin{align*}
  u_j(\bar{x}_i, y_{P_i}) & \leq u_j(\bar{x}), \quad \forall j \in P_i, \quad \forall y_{P_i} \in X_{P_i} \\
  u_j(\tilde{x}_i, \bar{y}_{P_i}) & \geq u_j(\tilde{x}), \quad \forall j \in P_i, \quad \forall \bar{y}_{P_i} \in X_{P_i}.
\end{align*}
\]  

(2.3)

Since the game is a zero-sum, then system (2.3) implies

\[
\begin{align*}
  & \sum_{j \in P_i} u_j(\bar{x}_i, y_{P_i}) \geq \sum_{j \in P_i} u_j(\bar{x}), \quad \forall y_{P_i} \in X_{P_i} \\
  & \sum_{j \in P_i} u_j(\tilde{x}_i, \bar{y}_{P_i}) \geq \sum_{j \in P_i} u_j(\tilde{x}), \quad \forall \bar{y}_{P_i} \in X_{P_i}.
\end{align*}
\]  

(2.4)

We have \(P = \{P_1, P_2\}\), then there exists \(P_k\) such that \(P_k = -P_i = P_{-i}\), taking into account from system (2.3), we obtain

\[
\begin{align*}
  u_j(\bar{x}_{P_i}, y_{-P_i}) & \leq u_j(\bar{x}), \quad \forall j \in -P_i, \quad \forall y_{-P_i} \in X_{-P_i} \\
  u_j(\tilde{x}_{P_i}, \bar{y}_{-P_i}) & \leq u_j(\tilde{x}), \quad \forall j \in -P_i, \quad \forall \bar{y}_{-P_i} \in X_{-P_i}.
\end{align*}
\]  

Thus,

\[
\begin{align*}
  & \sum_{j \in -P_i} u_j(\bar{x}_{P_i}, y_{-P_i}) \leq \sum_{j \in -P_i} u_j(\bar{x}), \quad \forall y_{-P_i} \in X_{-P_i} \\
  & \sum_{j \in -P_i} u_j(\tilde{x}_{P_i}, \bar{y}_{-P_i}) \leq \sum_{j \in -P_i} u_j(\tilde{x}), \quad \forall \bar{y}_{-P_i} \in X_{-P_i}.
\end{align*}
\]  

(2.5)

Let \(y_{P_i} = \bar{x}_{P_i}\) and \(\bar{y}_{P_i} = \bar{x}_{P_i}\) in (2.4). Thus, taking into account to system (2.5), we have

\[
\begin{align*}
  & \sum_{j \in -P_i} u_j(\bar{x}) \leq \sum_{j \in -P_i} u_j(\bar{x}_{P_i}, \bar{x}_{P_i}) \leq \sum_{j \in -P_i} u_j(\tilde{x}) \\
  & \sum_{j \in -P_i} u_j(\tilde{x}) \leq \sum_{j \in -P_i} u_j(\tilde{x}_{P_i}, \tilde{x}_{P_i}) \leq \sum_{j \in -P_i} u_j(\bar{x}).
\end{align*}
\]  

(2.6)

The system (2.6) implies that \(\sum_{j \in -P_i} u_j(\bar{x}) = \sum_{j \in -P_i} u_j(\tilde{x})\) and since \(i\) is any arbitrary element in \(\{1, 2\}\) and the game \(G\) (2.1) is a zero-sum, then we deduce \(\forall i \in \{1, 2\}, \sum_{j \in P_i} u_j(\bar{x}) = \sum_{j \in P_i} u_j(\tilde{x})\).
3 Main Results

This section first introduces the concept of $P$-consistence property, which is a key element in our first main result. The second result characterize the existence of a $P$-absolute optimal solution in compact and convex games.

**Definition 3.1 ($P$-consistence property)** Let $P = \{P_i\}_{i \in M}$ be a partition of $I$. A game $G (2.1)$ is said to satisfy the $P$-consistence property if for each $x \in X$, there exists $z \in X$ such that

$$z_S \in \bigcap_{j \in S} \text{Arg} \max \{u_j(x_{-S}, y_S), \ y_S \in X_S\}, \ \forall S \in P.$$  

**Remark 3.1** The $P$-consistence property is relatively easy to check. Indeed, the definition of the $P$-consistence property implies that $z_S$ is the maximum for the payoff function of each player in $S \in P$. Then, to check if the $P$-consistence property is satisfied is reduced to checking if every component $z_{h,S}$ of $z_j^{S}$ that is equal to $z_{h,S}$ of $z_j^{S}$ obtained for each $j_1, j_2 \in S$. If so, $z$ is a $P$-consistent coalition, i.e., $z \in X$ and $z_S \in \bigcap_{i \in S} \text{Arg} \max \{u_i(x_{-S}, y_S), \ y_S \in X_S\}$, which means the $P$-consistence property is satisfied.

**Definition 3.2 ($P$-Quasiconcave)** Let $P = \{P_i\}_{i \in M}$ be a partition of $I$. A game $G (2.1)$ is said to be $P$-quasiconcave if $X_i$ is convex $i \in I$ and for each $S \in P$, the function $y_S \rightarrow u_j(y_S, x_{-S})$ is quasiconcave on $X_S$, for each $j \in S$, and $x_{-S} \in X_{-S}$.

We then establish the following existence theorem on $P$-absolute solutions.

**Theorem 3.1** Suppose the $P = \{P_i\}_{i \in M}$ is a partition of $I$ and game $G (2.1)$ is compact, continuous, $P$-quasiconcave, and satisfies the $P$-consistence property. Then, it possesses a $P$-absolute solution.

**Proof.** Consider the following correspondence $C$ defined from $X$ in $X$ by

$$C_P(x) = \{z \in X : z_S \in \bigcap_{j \in S} \text{Arg} \max \{u_j(x_{-S}, y_S), \ y_S \in X_S\}, \ \forall S \in P\}.$$  

The $P$-consistence property, compactness, continuity and $P$-quasiconcavity of $G$ imply that for all $x \in X$, $C_P(x)$ is nonempty, closed convex valued and $C_P$ is upper hemicontinuous over $X$. Then, by Kakutani fixed point Theorem, there exists $\bar{x} \in X$ such that $\bar{x} \in C_P(\bar{x})$. Therefore, $\bar{x}_S \in \bigcap_{j \in S} \text{Arg} \max \{u_j(\bar{x}_{-S}, y_S), \ y_S \in X_S\}$, for all $S \in P$, which means $\bar{x}$ is $P$-absolute optimal solution. ■
**Remark 3.2** The following example shows that the $P$-consistence property is sufficient but not necessary.

**Example 3.1** Consider a game with $n = 3$, $I = \{1, 2, 3\}$, $X_1 = X_2 = [-1, 1]$ and

$$
\begin{align*}
    u_1(x) &= x_1 + x_2 x_3, \\
    u_2(x) &= x_1 + x_2 - x_3, \\
    u_3(x) &= x_1 - x_2 + x_3.
\end{align*}
$$

It can be easily seen that the game is compact, continuous and quasiconcave. Let us consider the following partition $P = \{\{1, 2\}, \{3\}\}$ of $\{1, 2, 3\}$. Then, the game considered possesses a $P$-absolute optimal solution that is $(1, 1, 1)$. However, it does not satisfy the $P$-consistence property. Indeed, let $x \in X$ such that $x_3 < 0$. Then, we obtain $\operatorname{Arg}\max\{u_1(y_1, y_2, x_3), \ y_1, 2 \in X_1, 2\} = \{(1, -1)\}$ and $\operatorname{Arg}\max\{u_2(y_1, y_2, x_3), \ y_1, 2 \in X_1, 2\} = \{(1, 1)\}$. Thus, $z_{1,\{1,2\}}^{1} = -1 \neq 1 = z_{2,\{1,2\}}^{2}$. Therefore, for each $x \in X$ with $x_3 < 0$, there does not exist $z \in X$ such that $z_{\{1,2\}} \in \bigcap_{j=1,2} \operatorname{Arg}\max\{u_j(y_1, 2, x_3), \ y_1, 2 \in X_1, 2\}$.

In order to characterize the existence of $P$-absolute optimal solutions of the game $G(2.1)$, we will use the following lemma due to Tian [1993].

Let us in first recall the definition of transfer continuity and transfer FS-convexity concepts introduced by Tian [1993].

**Definition 3.3 (Transfer Closed)** Let $X$ and $Y$ be two topological spaces. A correspondence $G : X \to 2^Y$ is said to be transfer closed-valued on $X$ if for every $x \in X$, $y \notin G(x)$ implies that there exists $x' \in X$ such that $y \notin \operatorname{cl}G(x')$.

**Definition 3.4 (Transfer FS-Convexity)** Let $X$ be a topological space and let $Y$ be a nonempty convex subset of a vector space $F$. A correspondence $G : X \to 2^Y$ is said to be transfer FS-convex on $X$ if for any finite subset $X^m = \{x^1, ..., x^m\} \in \langle X \rangle$, there exists a corresponding finite subset $Y^m = \{y^1, ..., y^m\} \in \langle Y \rangle$ such that for any subset $\{y^{k1}, y^{k2}, ..., y^{ks}\} \subset Y^m$, $1 \leq s \leq m$, we have $\operatorname{co}\{y^{k1}, y^{k2}, ..., y^{ks}\} \subset \bigcup_{j=1,...,s} G(x^{kj})$.

Tian [1993] has established the following lemma.

**Lemma 3.1** Let $X$ be a topological space and $Y$ be a nonempty compact convex subset in a Hausdorff topological vector space $F$. Suppose a correspondence $G : X \to 2^Y$ is transfer closed-valued and transfer FS-convex on $X$. Then, $\bigcap_{x \in X} G(x)$ is nonempty and compact.

---

*Tian [1993] characterizes the existence of greatest and maximal elements of weak and strict preferences. Conditions called transfer FS-convexity and transfer SS-convexity are shown to be necessary and, in addition with transfer closedness and transfer openness, sufficient for the existence of greatest and maximal elements of weak and strict preferences, respectively.*

---
Let us consider the following functions.

\[ \Psi_P : X \times \hat{X} \to \mathbb{R} \text{ defined as } (x, \hat{y}) \mapsto \Psi_P(x, \hat{y}) = \sum_{S \in P} \sum_{j \in S} u_j(x_{-S}, \hat{y}_S), \]

\[ g : X \to \hat{X} \text{ defined as } x \mapsto g(x) = (x_S, \ldots, x_S, S \in P), \]

where \( \#S \) is the number of elements in set \( S \) and \( \hat{X} = \prod_{S \in P} X_S^j \) and \( X_S^j = X_S, \forall j \).

**Remark 3.3** We have

\[ \forall x \in X, \quad \sup_{\hat{y} \in \hat{X}} \Psi_P(x, \hat{y}) \geq \Psi_P(x, g(x)), \]

due to if we put \( \hat{y} = g(x) \), and we obtain \( \Psi_P(x, \hat{y}) = \Psi_P(x, g(x)) \).

The following lemma gives the relation between \( P \)-absolute optimal solution of the game \( G \)

(2.1) and the function \( \Psi_P \).

**Lemma 3.2** The following two propositions are equivalent.

1. \( \bar{x} \in X \) is a \( P \)-absolute optimal solution of game (2.1).

2. \( \sup_{\hat{y} \in \hat{X}} \Psi_P(\bar{x}, \hat{y}) = \Psi_P(\bar{x}, g(\bar{x})). \)

**Proof.** Necessity \( (\Rightarrow): \) Let \( \bar{x} \in X \) be a \( P \)-absolute optimal solution of game \( G \) (2.1). Then

\[ u_j(\bar{x}_{-S}, \tilde{y}_S) \leq u_i(\bar{x}), \forall \bar{S} \in P, \forall t_i \in X_S \text{ and } \forall j \in S, \text{ hence } \Psi_P(\bar{x}, \tilde{y}) \leq \Psi_P(\bar{x}, g(\bar{x})), \forall \tilde{y} \in \hat{X} \]

i.e. \( \sup_{\tilde{y} \in \hat{X}} \Psi_P(\bar{x}, \tilde{y}) \leq \Psi_P(\bar{x}, g(\bar{x})). \) Taking into account of Remark 3.3, we obtain

\[ \sup_{\tilde{y} \in \hat{X}} \Psi_P(\bar{x}, \tilde{y}) = \Psi_P(\bar{x}, g(\bar{x})). \]

Sufficiency \( (\Leftarrow): \) Let \( \bar{x} \in X \) such that \( \sup_{\tilde{y} \in \hat{X}} \Psi_P(\bar{x}, \tilde{y}) = \Psi_P(\bar{x}, g(\bar{x})), \) this equality implies

\[ \forall \tilde{y} \in \hat{X}, \quad \sum_{S \in P} \sum_{j \in S} \{u_j(\bar{x}_{-S}, \hat{y}_S) - u_j(\bar{x})\} \leq 0. \]

For some fixed \( \bar{S} \in P \) and \( i \in S \), we have \( \forall \tilde{y} \in \hat{X} \),

\[ u_i(\bar{x}_{-S}, \hat{y}_S) - u_i(\bar{x}) + \sum_{j \in S \setminus \{i\}} \{u_j(\bar{x}_{-S}, \hat{y}_S) - u_j(\bar{x})\} + \sum_{K \in P \setminus \{S\}} \sum_{j \in K} \{u_j(\bar{x}_{-K}, \hat{y}_K) - u_j(\bar{x})\} \leq 0. \]

For \( \tilde{y} \in \hat{X} \) such that \( \hat{y}_S = (\hat{y}_S^1, \bar{x}_S, \ldots, \bar{x}_S) \) with \( \hat{y}_S^1 \) is arbitrarily chosen in \( X_S \) and \( \hat{y}_K = (\bar{x}_K, \ldots, \bar{x}_K), \forall K \neq S \), we obtain then

\[ \sum_{j \in S \setminus \{i\}} \{u_j(\bar{x}_{-S}, \hat{y}_S) - u_j(\bar{x})\} + \sum_{K \in P \setminus \{S\}} \sum_{j \in K} \{u_j(\bar{x}_{-K}, \hat{y}_K) - u_j(\bar{x})\} = 0 \]
Then from the last inequality, we deduce that $\forall \tilde{y} \in X_S$, $u_i(\pi_{-S}, \tilde{y}) \leq u_i(\pi)$, since $S$ and $i$ are arbitrarily chosen in $P$, $S$, respectively. Thus we have,

$$\forall \tilde{y} \in X_S, \ u_i(\pi_{-S}, \tilde{y}) \leq u_i(\pi),$$

for each $S \in P$ and $i \in S$, therefore $\pi$ is a $P$-absolute optimal solution of the game $G$ (2.1). This completes the proof. ■

Lemma 3.2 transforms the problem of determination of $P$-absolute optimal solutions of the game $G$ (2.1) into a problem of determination of issues $\pi \in X$ verifying

$$\sup_{\tilde{y} \in X} \{\Psi_P(\pi, \tilde{y}) - \Psi_P(\pi, g(\pi))\} = 0.$$

Let us consider the following definitions.

**Definition 3.5** Let $Z$ be a nonempty subset of a topological space, $Y$ be a nonempty set and let $g : Z \rightarrow Y$ be a function. A function $f : Z \times Y \rightarrow \mathbb{R}$ is said to be $g$-diagonally transfer lower continuous in $x$ if for every $(x, y) \in Z \times Y$, $f(x, y) > f(x, g(x))$ implies that there exist some point $y' \in Y$ and some neighborhood $\mathcal{V}(x) \subset Z$ of $x$ such that $f(z, y') > f(z, g(z))$ for all $z \in \mathcal{V}(x)$.

$g$-Diagonally transfer lower continuity in $x$ says that if a point $x$ in $X$ is upset by a deviation point $y$ in $Y$ comparing to $f(x, g(x))$, then there is an open set of points containing $x$, all of which can be upset by a single deviation point $y'$. Here, transfer lower continuity in $x$ refers to the fact that $y$ may be transferred to some $y'$ in order for the inequality to hold for all points in a neighborhood of $x$ and $g$-diagonal refers to the point $(x, g(x))$. It is clear that if the function $f(x, y) - f(x, g(x))$ is lower semi-continuous in $x$, then it is also $g$-diagonally transfer lower continuous in $x$, but the converse is not true.

**Definition 3.6** Let $Z$ be a nonempty convex subset of a vector space $E$, let $Y$ be a nonempty set and let $g : Z \rightarrow Y$ be a function. A function $f : Z \times Y \rightarrow \mathbb{R}$ is said to be $g$-diagonally transfer quasiconcave in $y$ on $Y$ if, for any finite subset $\{y^1, ..., y^m\} \subset Y$, there exists a corresponding finite subset $\{x^1, ..., x^m\} \subset Z$ such that for any subset $J \subset \{1, 2, ..., m\}$, and any $x \in co\{x^h, \ h \in J\}$ we have $\min_{h \in J} f(x, y^h) \leq f(x, g(x))$.

$g$-Diagonally transfer quasiconcavity roughly says that given any finite subset $Y^m = \{y^1, ..., y^m\}$ of deviation profiles, there exists a corresponding finite subset $X^m = \{x^1, ..., x^m\}$ of candidate profiles such that for any subset $\{x^{k_1}, x^{k_2}, ..., x^{k_s}\} \subset X^m$, $1 \leq s \leq m$, so that its convex combinations are not upset by those deviation profiles in $X^m$.

The following theorem characterizes the existence of absolute optimal solution in a compact and convex game.
**Theorem 3.2** Let \( X = \prod_{i \in I} X_i \) be a nonempty compact, convex subset of a Hausdorff locally convex vector space \( \mathbb{R}^{(i)} \). Assume that \( \Psi_P(x, \hat{y}) \) is a \( g \)-diagonally transfer lower continuous in \( x \). Then, the game \( G(2.1) \) has at least one \( P \)-absolute optimal solution if and only if the function \( \Psi_P(x, \hat{y}) \) is \( g \)-diagonally transfer quasiconcave in \( \hat{y} \).

**Proof.** Necessity (\( \Rightarrow \)): Let \( \pi \in X \) be the \( P \)-absolute optimal solution of game \( G(2.1) \). Then, Lemma 3.2 implies that
\[
\sup_{\hat{y} \in \hat{X}} \Psi_P(\pi, \hat{y}) = \Psi_P(\pi, g(\pi)).
\]

Let \( B = \{\hat{y}^1, ..., \hat{y}^m\} \subset \hat{X} \). Then, there exists a corresponding finite subset \( A = \{x^1, ..., x^m\} \subset X \) where \( x^j = \pi \), \( j = 1, ..., m \) such that every \( J \subset \{1, ..., m\} \), and all \( x \in \text{co}\{x^h, h \in J\} \), we have
\[
\min_{h \in J} \Psi_P(\pi, \hat{y}^h) \leq \sup_{\hat{y} \in \hat{X}} \Psi_P(\pi, \hat{y}) = \Psi_P(\pi, g(\pi)).
\]

**Sufficiency (\( \Leftarrow \)):** Let us consider the following collection
\[
G(\hat{y}) = \{x \in X, \; \Psi_P(x, \hat{y}) \leq \Psi_P(x, g(x))\}, \; \hat{y} \in \hat{X}.
\]

We have the \( g \)-diagonally transfer lower semicontinuous in \( x \) and \( g \)-diagonally transfer quasiconcave in \( \hat{y} \) of \( \Psi_P(x, \hat{y}) \) implies that the collection \( G(\hat{y}) \) is transfer closed-valued and FS-convex on \( X \). Then, from Lemma 3.3, we conclude \( \bigcap_{\hat{y} \in \hat{X}} G(\hat{y}) \) is nonempty, i.e. there exists \( \pi \in X \) such that \( \pi \in G(\hat{y}) \), for all \( \hat{y} \in \hat{X} \). Hence, \( \sup_{\hat{y} \in \hat{X}} \Psi_P(\pi, \hat{y}) = \Psi_P(\pi, g(\pi)) \). From Lemma 3.2, \( \pi \) is a \( P \)-absolute optimal solution of the game \( G(2.1) \). This completes the proof. \( \blacksquare \)

If the partition \( \{P_i\}_{i \in I} \) is reduced to the following \( \{i\}_{i \in I} \), then Theorem 3.2 is identical to the Theorem 1 of Baye et al. [1993].

Taking into account Remark 3.3 and Lemma 3.2, we obtain the following proposition.

**Proposition 3.1** The game \( G(2.1) \) has at least one \( P \)-absolute optimal solution if and only if
\[
\alpha = 0,
\]
where
\[
\alpha = \inf_{x \in X} \sup_{\hat{y} \in \hat{X}} \{\Psi_P(x, \hat{y}) - \Psi_P(x, g(x))\}. \tag{3.1}
\]

From this Proposition we obtain the following method for the determination of \( P \)-absolute optimal solutions of game \( G(2.1) \).

We deduce the following corollary of \( g \)-maximum inequality.
Algorithm 1. Procedure for the determination of a $P$-absolute optimal solution

**Require:** Calculate the value $\alpha = \inf_{x \in X} \sup_{\hat{y} \in \hat{X}} \{\Psi_P(x, \hat{y}) - \Psi_P(x, g(x))\}$ of (3.1).

if $\alpha > 0$, then

the game $G$ (2.1) has no $P$-absolute optimal solution.

else

any strategy profile $\pi \in X$ such that $\sup_{\hat{y} \in \hat{X}} \Psi_P(\pi, \hat{y}) = \Psi_P(\pi, g(\pi))$ are $P$-absolute optimal solutions of the game $G$ (2.1).

end if

**Corollary 3.1** Let $X$ be a nonempty compact, convex subset of a Hausdorff locally convex vector space $E$, let $Y$ be a nonempty set and $g : X \to Y$ be a function. Let $\Phi$ be a real-valued function on $X \times Y$ such that $\Phi(x, y)$ is a $g$-diagonally transfer lower semicontinuous in $x$. Then, the function $\Phi(x, y)$ is $g$-diagonally transfer quasiconcave in $y$ if and only if there exists $\pi \in X$, such that

$$\Phi(\pi, y) \leq \Phi(\pi, g(\pi)), \quad \forall y \in Y.$$  

### 3.1 Extension to Multiple Objectives

Consider the following $n$-person multiple objectives game in normal form defined as

$$MOG = \langle I, X^i, u^i \rangle$$

(3.2)

where $I = \{1, \ldots, n\}$ is the set of players, $X = \prod_{i \in I} X^i$ is the set of strategies of the game, where $X^i$ (a nonempty set is in $\mathbb{R}^{l(i)}$) is the set of strategies of player $i$ and $u^i$ is $i$’s payoff function $X \to \mathbb{R}^{m(i)}$.

In MOG (3.2), each player has a vector payoff to optimize, so if $m(i) = 1$ for each player, then the MOG (3.2) becomes exactly an $n$-person game $G$ defined in (2.1). In the case, where $n = 1$ and $m(1) > 1$, MOG (3.2) becomes the standard *multiple objectives mathematical programming* (MOMP)\(^\text{7}\) problem.

For each player $i$ in $I$, let

$$\Delta_i = \{\lambda_i = (\lambda_{i1}^1, \ldots, \lambda_{im(i)}^i) \in \mathbb{R}^{m(i)} : \lambda_{ij}^i \geq 0, \forall j = 1, \ldots, m(i) \text{ and } \sum_{j=1}^{m(i)} \lambda_{ij}^i = 1\}$$

be the unit simplex of $\mathbb{R}^{m(i)}$.

\(^7\)For more details about the MOMP problem, see Zhao [1983]
**Definition 3.7** Let \( i \) be any player in \( I \). The following function is said to be \( \lambda_i \)-**weighted payoff** function of player \( i \):

\[
WU_{\lambda_i,i} : X \rightarrow \mathbb{R} \text{ defined by } x \mapsto WU_{\lambda_i,i}(x) = \sum_{j=1}^{m(i)} \lambda_{i,j} u_j^i(x) \text{ with } \lambda_i \in \Delta_i.
\]

Zhao [1983] defines the concepts of hybrid solution and quasi-hybrid solution for MOG (3.2), and he shows that under primitive conditions, MOG (3.2) possesses hybrid and quasi-hybrid solutions.

**Definition 3.8** Let \( P = \{P_i\}_{i \in M} \) be a partition of \( I \). A feasible strategy \( \pi \in X \) is said to be an **hybrid solution** (HS) for the game MOG (3.2) if for each coalition \( S \in P \), \( \pi_S \) is a core solution \( MOG_S(\pi_{-S}) \), where

\[
MOG_S(\pi_{-S}) = \langle S, X^i, u_i(x_S, \pi_{-S}) \rangle.
\]

**Definition 3.9** Let \( P = \{P_i\}_{i \in M} \) be a partition of \( I \). A feasible strategy \( \pi \in X \) is said to be an **quasi-hybrid solution** (QHS) for the game MOG (3.2) if for each coalition \( S \in P \), \( \pi_S \) is a properly Pareto efficient strategy profile of the following vector maximization problem \( VM_S(\pi_{-S}) \) defined by

\[
VM_S(\pi_{-S}) : \left\{ \max_{x_S \in X_S} u_S(x_S, \pi_{-S}) = [u_i(x_S, \pi_{-S})], \ i \in S \right\}.
\]

We introduce the following definition of weighted absolute optimal solution.

**Definition 3.10** Let \( P = \{P_i\}_{i \in M} \) be a partition of \( I \). A feasible strategy \( \bar{\pi} \in X \) is said to be a **P-weighted absolute optimal solution** (WAO) relative to \( \lambda \in \Delta = \prod_{i \in I} \Delta_i \) for the game MOG (3.2) if for each coalition \( S \in P \), we have

\[
WU_{\lambda_S,S}(\bar{\pi}) \succeq WU_{\lambda_S,S}(y_S, \pi_{-S}), \ \forall y_S \in X_S,
\]

where \( WU_{\lambda_S,S}(x) = (WU_{\lambda_i,i}(x), \ i \in S) \). Or precisely

\[
\bar{\pi}_S \in \bigcap_{j \in S} \text{Argmax} \{WU_{\lambda_j,j}(y_S, \pi_{-S}), \ y_S \in X_S\} \text{ for all } S \in P.
\]

A strategy profile \( \bar{\pi} \) is a **P-weighted absolute optimal solution** if no player in any coalition \( S \) in \( P \), can be better off when the players deviate from their **P-weighted absolute optimal strategy profile** \( \bar{\pi}_S \). This means that at a **P-weighted absolute optimal**, the players in coalition \( S \) play a strategy profile that maximizes the weighted payoffs.

**Property 3.1** Any **P-absolute optimal solution** is also a **P-weighted absolute optimal solution**. And if for each \( i \in I, m(i) = 1 \), then the two solutions are identical.
We can also define a weighted strong Nash equilibrium relative to partition $P$ as follows.

**Definition 3.11** A strategy profile $\pi \in X$ is said to be $P$-weighted properly strong Nash equilibrium (PWSNE) relative to $\lambda \in \Delta$ of game MOG (3.2), if for each $S \in P$ and $K \in \Im S$, $\pi_K \in X_S$ is properly Pareto efficient of the vector maximization problem:

$$WVM_K(\pi_K) : \left\{ \max_{x_K \in X_K} WU_{\lambda,K}(x_K, \pi_K) = \left[WU_{\lambda,i}(x_K, \pi_K), \ i \in K \right] \right\}.$$ 

**Property 3.2** Any $P$-weighted absolute optimal solution is also a $P$-weighted properly strong Nash equilibrium.

**Definition 3.12** ($P$-Weighted Consistency Property) Let $P = \{P_i\}_{i \in M}$ be a partition of $I$. A game MOG (3.2) is said to satisfy the $P$-weighted consistency property relative to $\lambda \in \Delta$ if for each $x \in X$, there exists $z \in X$ such that

$$z_S \in \bigcap_{j \in S} \text{Arg} \max \{WU_{\lambda,j}(x_S, y_S), \ y_S \in X_S\}, \ \forall S \in P.$$ 

**Definition 3.13** ($P$-Weighted Quasiconcave) Let $P = \{P_i\}_{i \in M}$ be a partition of $I$. A game MOG (3.2) is said to be $P$-weighted quasiconcave relative to $\lambda \in \Delta$ if $X^i$ is convex, $i \in I$ and for each $S \in P$, the function $y_S \to WU_{\lambda,P}(y_P, x_P)$ is quasiconcave on $X_S$, for each $i \in S$, and $x_P \in X_{P}$. 

We then establish the following existence theorem on $P$-weighted absolute solutions.

**Theorem 3.3** Suppose the $P = \{P_i\}_{i \in M}$ is a partition of $I$ and game MOG (3.2) is compact, continuous, $P$-weighted quasiconcave, and satisfies the $P$-weighted consistency property relative to $\lambda \in \Delta$. Then, it possesses a $P$-weighted absolute solution relative to $\lambda$.

**Proof:** The proof of this theorem is similar to that of Theorem 3.1.

In the same manner, we can characterize the existence of a $P$-weighted absolute optimal solution in compact and convex games. Let us consider the following functions.

$$\Psi_{\lambda,P} : X \times \hat{X} \to \mathbb{R} \text{ defined as } (x, \hat{y}) \mapsto \Psi_{\lambda,P}(x, \hat{y}) = \sum_{S \in P} \sum_{i \in S} \sum_{j=1}^{m(i)} \lambda^j_i u^j_i(x_S, y_S)$$

$$g : X \to \hat{X} \text{ defined as } x \mapsto g(x) = (x_S, \ldots, x_S, S \in P),$$

where $\hat{X} = \prod_{S \in P} \prod_{j \in S} X^j_S$ and $X^j_S = X_S, \forall j$.

We have the following theorem.

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8We can also define the $P$-weighted super-strong Nash equilibrium, $P$-weighted strong Nash equilibrium by replacing the “properly Pareto efficient” by “Pareto efficient”, and “weakly Pareto efficient” in Definition 3.11, respectively.
**Theorem 3.4** Let $X = \prod_{i \in I} X_i$ be a nonempty compact, convex subset of a Hausdorff locally convex vector space $\mathbb{R}^{l(i)}$. Assume that $\Psi_{\lambda,P}(x, \hat{y})$ is a $g$-diagonally transfer lower continuous in $x$. Then, the game MOG (3.2) has at least one $P$-weighted absolute optimal solution relative to $\lambda$ if and only if the function $\Psi_{\lambda,P}(x, \hat{y})$ is $g$-diagonally transfer quasiconcave in $\hat{y}$.

**Proof.** The proof of this theorem is similar to that of Theorem 3.2. 

## 4 Conclusion

There is no general theorem on the existence of absolute optimal solutions in the literature. In this paper, we fill this gap by proposing some existence results on absolute optimal solutions relative to a partition of players $P$ in general $n$-person and multiple objectives games. We provide a condition (called $P$-consistency property) which together with the $P$-quasiconcavity and continuity of payoffs permits the existence of $P$-absolute optimal solution in $n$-person games with compact and convex strategy spaces. The $P$-consistency property is a general condition that cannot be dispensed with for the existence of an absolute optimal solution. Furthermore, it is relatively easy to check. We also characterize the existence of a $P$-absolute optimal solution by providing a necessary and sufficient condition. Moreover, we suggest a procedure that can be used to efficiently compute $P$-absolute optimal solutions.
References


